Fourier Transform

The Infinite fourier transform (or) complex fourier transform of a function $f(x)$ is define by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= F(s)$$

The function $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)] e^{-isx} ds$

is called the inverse formulea for the fourier transform and it is denoted by $F^{-1}[F(f(x))]$

Probl Find the Fourier transform of $f(x)$

$$f(x) = \begin{cases} x & |x| < a \\ 0 & |x| > a \end{cases}$$

Soln:-

given $f(x) = x, \quad -a \leq x \leq a$

$f(x) = 0 \quad -\infty < x < -a$ &

$a < x < \infty$

The fourier transform $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} f(x) e^{isx} dx + \int_{-a}^{a} f(x) e^{isx} dx + \int_{a}^{\infty} f(x) e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} x\, e^{isx}\, dx\right]$$ ②

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x\,(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} x \cos sx\, dx + i\int_{-a}^{a} \sin sx\; x\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[i\,2\int_{0}^{a} x \sin sx\, dx\right]$$ [$\because$ Since $x \cos sx$ is a odd fun]

$$= \frac{\sqrt{2}\, i}{\sqrt{\pi}} \int_{0}^{a} x \sin sx\, dx.$$

$\int u v\, dx$

$\cancel{u} = x \qquad dv = \sin sx$

$$\int u v\, dv = u v_1 - u' v_2 + u'' v_3 - \cdots$$

$u = x \qquad v = \sin sx$

$u' = 1 \qquad v_1 = -\frac{\cos sx}{s}$

$u'' = 0.$

$v_2 = -\frac{\sin sx}{s^2}$

$$= \frac{1}{\cancel{\sqrt{2\pi}}} = \frac{\sqrt{2}\, i}{\sqrt{\pi}}\left[x\left(-\frac{\cos sx}{s}\right) - 1\left(-\frac{\sin sx}{s^2}\right)\right]_0^a$$

$$= \frac{\sqrt{2}\, i}{\sqrt{\pi}}\left\{\left[\frac{-a \cos as}{s} + \frac{\sin sa}{s^2}\right] - \left[0 + 0\right]\right\}$$

$$= \frac{\sqrt{2}i}{\sqrt{\pi}} \left\{ \frac{\sin sa}{s^2} - \frac{a\cos sa}{s} \right\}$$ (3)

$$= \sqrt{\frac{2}{\pi}}\, i \left[\frac{\sin sa - as\cos sa}{s^2} \right]$$

Prob 2 Find the fourier transform of

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

Soln:-

$f(x) = 1 \qquad -a \le x \le a$

$f(x) = 0 \qquad -\infty < x < -a$ &

$a < x < \infty$

$$\text{The fourier transform} = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} f(x)\, e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} f(x)\, e^{isx} dx + \int_{-a}^{a} f(x)\, e^{isx} dx + \int_{a}^{\infty} f(x)\, e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} e^{isx} dx \right] \quad \because \text{Since } (f(x) = 0)$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{a}$$

$$\cos x = \frac{e^{ix} + e^{-i}}{2}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{is} \left[e^{isa} - e^{-isa} \right] \qquad \sin x = \frac{e^{ix} - e}{2}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{is} \left[2i \sin as \right]$$

$= \sqrt{2/\pi}\left(\frac{\sin as}{s}\right)$

Prob 3 Find $\mathcal{F}(f(x))$ define by

$$f(x) = \begin{cases} x^2 & |x| \le a \\ 0 & |x| > a \end{cases}$$ [$x^2 \cos x$ is a even function]

definition Self reciprocal

If $\mathcal{F}[f(x)]$ is $f(s)$ then $f(x)$ is Self Reciprocal under fourier transform.

Prob 4 S.T The fourier transform of $f(x)$

$f(x) = e^{-x^2/2}$ is $e^{-s^2/2}$.

(or)

$e^{-x^2/2}$ is self reciprocal under fourier transform.

Soln :-

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, e^{isx}\, e^{s^2/2}\, e^{-s^2/2}\, dx$$

$$= \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2 + isx + s^2/2}\, dx.$$

$$= \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(x^2 - 2isx - s^2)} dx.$$

$$= \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(x - is)^2} dx.$$

Put $u = \frac{x - is}{\sqrt{2}} \Rightarrow u^2 = \frac{(x-is)^2}{2}.$

$du = 1/\sqrt{2}\,(dx - 0)$

$dx = \sqrt{2}\, du$

$x = -\infty \Rightarrow u = -\infty$

$x = \infty \Rightarrow u = \infty$

$$\frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \sqrt{2}\, du$$

$$= \frac{e^{-s^2/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du \qquad (e^{-u^2} \text{ is a even fun})$$

$$= \frac{e^{-s^2/2}}{\sqrt{\pi}}\, 2 \int_{0}^{\infty} e^{-u^2} du \qquad \left(\because \int_0^\infty e^{-x^2} du = \sqrt{\pi}/2\right)$$

$$= \frac{e^{-s^2/2}}{\cancel{\sqrt{\pi}}}\, \cancel{2}\, \cancel{\sqrt{\pi}}/\cancel{2}$$

$$= \boxed{e^{-s^2/2}}$$

Prob 3 Find $F[f(x)]$

$$f(x) = \begin{cases} x^2 & |x| \le a \\ 0 & |x| > a \end{cases}$$

$$f(x) = x^2 \quad -a \le x \le a$$

$$f(x) = 0 \quad -\infty \le x \le -a$$

$$a \le x \le \infty.$$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} f(x)\, e^{isx}\, dx + \int_{-a}^{a} f(x)\, e^{isx}\, dx + \int_{a}^{\infty} f(x)\, e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} x^2 e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} x^2 (\cos sx + i \sin sx)\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x^2 \cos sx + i \sin sx\, x^2\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x^2 \cos sx\, dx + \int_{-a}^{a} x^2 i \sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x^2 \cos sx \, dx \qquad \because \left[x^2 \sin sx \text{ is a odd fun} \right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} x^2 \cos sx \, dx \qquad \left[x^2 \cos sx \text{ is a even fun} \right]$$

$$= \sqrt{2/\pi} \int_{0}^{a} x^2 \cos sx \, dx \quad \text{——} \; (1)$$

$u = x^2 \qquad dv = \cos sx$

$u' = 2x \qquad v = \frac{\sin sx}{s}$

$u'' = 2 \qquad v_1 = -\frac{\cos sx}{s^2}$

$v_2 = -\frac{\sin sx}{s^3}$

Consider

$$\int u \, dv = \left[uv - u'v_1 + u''v_2 - \cdots \right]$$

$$\int_0^a x^2 \cos sx \, dx = \left[x^2\left(\frac{\sin sx}{s}\right) - 2x\left(-\frac{\cos sx}{s^2}\right) + 2\left(-\frac{\sin sx}{s^3}\right) \right]_0^a$$

$$= \left[\frac{a^2 \sin sa}{s} + \frac{2a \cos sa}{s^2} - \frac{2 \sin sa}{s^3} \right]$$

$$(1) \Rightarrow \sqrt{2/\pi} \left\{ \frac{(as)^2 \sin sa + 2as \cos sa - 2 \sin sa}{s^3} \right\}$$

Prob 5 Find the Fourier transform of ⑧

$$f(x) = \begin{cases} e^{ikx} & a < x < b \\ 0 & x < a,\ x > b \end{cases}$$

Soln:-

$$f(x) = e^{ikx} \qquad a < x < b$$

$$f(x) = 0 \qquad -\infty < x < a,\ b < x < \infty$$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{a} f(x)\, e^{isx}\, dx + \int_{a}^{b} f(x)\, e^{isx}\, dx + \int_{b}^{\infty} f(x)\, e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{a}^{b} f(x)\, e^{ikx} e^{isx}\, dx. \right.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{ix(s+k)}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(k+s)x}}{i(k+s)} \right]_{a}^{b}$$

$$= \frac{1}{\sqrt{2\pi}(k+s)i} \left[e^{i(k+s)b} - e^{i(k+s)a} \right]$$

$$= \frac{-i}{\sqrt{2\pi}(k+s)} \left[e^{i(k+s)b} - e^{i(k+s)a} \right]$$

$$= \frac{i}{\sqrt{2\pi}(k+s)}\left[e^{i(k+s)a} - e^{i(k+s)b}\right]$$ (4)

Prob 6 Find the fourier transform of

$$f(x) = \begin{cases} \cos x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Soln

$$f(x) = \cos x \qquad 0 < x < 1$$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} f(x)\, e^{isx} dx + \int_{0}^{1} f(x)\, e^{isx} dx + \int_{1}^{\infty} f(x)\, e^{isx} dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{0}^{1} \cos x\; e^{isx} dx\right] \qquad \begin{matrix} a = is \\ b = 1 \end{matrix}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{(is)^2 + 1^2}\right]_0^1 \qquad \int e^{ax}\cos bx\, dx$$

$$\left[is\cos x + \sin x\right]_0^1 \qquad = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{is}}{1-s^2}\left[(is\cos 1 + \sin 1)\right.\right.$$
$$\left.\left. \frac{e^0}{1-s^2}(-is\cos 0 + \sin 0)\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[\frac{e^{is}}{1-s^2}(is\cos 1 + \sin 1) - \frac{1}{1-s^2}(is)\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{1-s^2} \left[e^{is} (is\cos 1 + \sin 1) - is \right] \right.$$

Prob 7 Find the fourier transform of

$$\frac{1}{\sqrt{|x|}}$$

Soln

$$F(x)\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} (\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} \cos sx\, dx + \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}}\, i\sin sx\, dx \right.$$

$\left[\frac{1}{|x|} \cos sx \text{ is a even function} \right]$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} \cos sx\, dx = 2\int_{0}^{\infty} \frac{1}{\sqrt{|x|}} \cos sx\, dx.$$

$\left[\frac{1}{\sqrt{|x|}}\, i\sin sx\, dx \text{ is a Odd function} \right]$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}}\, i\sin sx\, dx = 0.$$

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$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \frac{\cos sx}{\sqrt{|x|}} dx$$

$$= \sqrt{2/\pi} \int_0^{\infty} \frac{\cos sx}{\sqrt{x}} dx \quad \left[\because (0,\infty) \; |x| = x \right]$$

$$= \sqrt{2/\pi} \int_0^{\infty} x^{-1/2} \cos sx \, dx \quad \text{formula}$$

$\sin(n\pi/2) \cdot \Gamma(n)$

$$= \sqrt{2/\pi} \quad \left[\because \int_0^{\infty} \cos sx \, x^{n-1} dx = \frac{\cos n\pi/2}{s^n} \Gamma n \right]$$

$$= \sqrt{2/\pi} \left[\frac{\cos \pi/4}{s^{1/2}} \Gamma\tfrac{1}{2} \right] \sqrt{\pi}$$

$\Gamma m+n$

$$= \left[\sqrt{2/\pi} \; \frac{1}{\sqrt{2}} \; \frac{1}{\sqrt{s}} \; \sqrt{\pi} \right]$$

$$= \frac{1}{\sqrt{s}} \; //$$

Prob 8 Find the fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 0, & x < a \\ 1 & a < x < b \\ 0 & x > b. \end{cases}$$

Ans.

Soln:-

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{a} f(x) e^{isx} dx + \int_{a}^{b} f(x) e^{isx} dx + \int_{b}^{\infty} f(x) e^{isx} dx \right]$$

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$$= \frac{1}{\sqrt{2\pi}}\left[\int_a^b e^{isx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isb}}{is} - \frac{e^{isa}}{is}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isb} - e^{isa}}{is}\right]$$

Prob 9 Find the fourier trans form of

$$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & \text{Otherwise.} \end{cases}$$

$\cos\pi = -1$
$\sin\pi = 0$

Soln:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} f(x)\, e^{isx}\,dx + \int_{0}^{\pi} f(x)\, e^{isx}\,dx + \int_{\pi}^{\infty} f(x)\, e^{isx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\pi} \sin x\; e^{isx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{(is)^2+1^2}\left(is\,\cancel{\sin x} - \cancel{\sin x}\right)\right]_0^{\pi}$$

$\cos x$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{1-s^2}\left(is\sin x - \cos x\right)\right]_0^{\pi}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{is\pi}}{1-s^2}\left(is\sin\pi - \cos\pi\right)\right]$$

$$\left[\frac{1}{1-s^2}(0-1)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{is\pi}}{1-s^2}(0+1)\right] + \frac{1}{1-s^2}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{is\pi}}{1-s^2} + \frac{1}{1-s^2}\right]$$

$$= \frac{1+e^{is\pi}}{\left(\sqrt{2\pi}\right)\left(1-s^2\right)} \;//$$

Prob 10

$$f(x) = \begin{cases} 1+x/a & -a<x<0 \\ 1-x/a & 0<x<a \\ 0 & \text{otherwise} \end{cases}$$

Soln:

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} f(x) e^{isx} dx + \int_{-a}^{0} f(x) e^{isx} dx + \int_{0}^{a} f(x) e^{isx} + \int_{a}^{\infty} f(x) e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{0} (1+x/a) e^{isx} dx + \int_{0}^{a} (1-x/a) e^{isx} dx \right] \quad \text{---} (1)$$

Consider,

$u = \left(\frac{a+x}{a}\right)$ $\quad v = e^{isx}$

$u' = \frac{1}{a}(1)$ $\quad v_1 = \frac{e^{isx}}{is}$

$u'' = 0$ $\quad v_2 = \frac{e^{isx}}{-s^2}$

$$\int_{-a}^{0} (1+x/a) e^{isx} dx = \left[\left(\frac{a+x}{a}\right)\left(\frac{e^{isx}}{is}\right) - \left(\frac{1}{a}\frac{e^{isx}}{-s^2}\right) \right]_{-a}^{0}$$

$$= \left[\left(\frac{1}{is}\right) + \left(\frac{1}{as^2}\right) - \frac{e^{-isa}}{as^2} \right]$$

$$= \left[\frac{as + i + ie^{-isa}}{ias^2}\right]$$

Now

$$u = \frac{a-x}{a} \qquad V = e^{isx}$$

$$u' = \frac{1}{a}(-1) \qquad V_1 = \frac{e^{isx}}{is}$$

$$u'' = 0 \qquad V_2 = \frac{e^{isx}}{-s^2}$$

$$\int_0^a \left(1 - \frac{x}{a}\right) e^{isx}\,dx = \left[\left(\frac{a-x}{a}\right)\left(\frac{e^{isx}}{is}\right) + \left(\frac{1}{a}\right)\left(\frac{e^{isx}}{-s^2}\right)\right]_0^a$$

$$= \left[0\left(\frac{e^{isa}}{is}\right) + \left(\frac{1}{a}\right)\frac{e^{isa}}{-s^2} - \left(\frac{1}{is}\right) - \left(\frac{1}{a}\right)\frac{1}{-s^2}\right]$$

$$= \left[-\frac{e^{isa}}{as^2} - \frac{1}{is} + \frac{1}{as^2}\right]$$

(I) =)

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{is} + \frac{1}{as^2} - \frac{1}{as^2}e^{isa} + \frac{1}{as^2} - \frac{1}{is} - \frac{1}{as^2}e^{isa}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[2/as^2 - \frac{1}{as^2}\left(e^{isa} + e^{isa}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{as^2}\left[2 - \left(e^{isa} + e^{isa}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{as^2}\left[2 - 2\cos as\right]$$

$$= \frac{1}{\sqrt{2\pi}}\frac{2}{as^2}\left[1 - \cos as\right]$$

$$= \frac{1}{\sqrt{2\pi}}\frac{2}{as^2}\left[2\sin^2\left(\frac{as}{2}\right)\right]$$

$$= \frac{4\sin^2(as/2)}{\sqrt{2\pi}\, as^2}$$

(II) (i) $e^{-a^2x^2}$, $a>0$

Hence show that $e^{-x^2/2}$ is self reciprocal of under fourier transform.

Soln:-

$$\mathcal{F} f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2+isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2-isx)}\, dx.$$

W.k.t $(b-c)^2 = b^2 - 2bc + c^2$

$$b = ax$$

$$2bc = isx$$

$$c = is/2a$$

$$b^2 - 2bc = (b-c)^2 - c^2.$$

$$a^2x^2 - 2ax(is/2a) = (ax - is/2a)^2 - (is/2a)^2$$

$$a^2x^2 - isx = \left[(ax - is/2a)^2 + \frac{s^2}{4a^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2-isx)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(ax-is/2)^2 + \frac{s^2}{4a^2}\right]} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(ax-is/2a)^2 - \frac{s^2}{4a^2}} dx.$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-(ax-is/2a)^2} dx.$$

Put $ax - is/2a = t$.

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$a \cdot dx = dt$ $\quad x = -\infty \Rightarrow t = -\infty$

$x = \infty \Rightarrow t = \infty$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-t^2} dt/a$$

$$= \frac{1}{a\sqrt{2\pi}} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-t^2} dt \Rightarrow \sqrt{\pi}$$

$$= \frac{1}{a\sqrt{2\pi}} e^{-s^2/4a^2} (\sqrt{\pi})$$

$$F[f(x)] = \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$

Put $a = 1/\sqrt{2}$

$$\frac{e^{-s^2/4(1/2)}}{}$$

$$= e^{-s^2/2}$$

$$\mathcal{F}\left[e^{-x^2/2}\right] = e^{-s^2/2}$$

$$\mathcal{F}\left[e^{-x^2/2}\right] = e^{-s^2/2}$$

(12) $f(x) = xe^{-x} \quad 0 \le x \le \infty$ $\quad -e^{-x}(x+1)+$

$$\frac{1}{\sqrt{2\pi}} \frac{1}{(1-is)^2}$$

(13). $f(x) = \begin{cases} \dfrac{\sqrt{2\pi}}{2\epsilon}, & |x| \le \epsilon \\ 0 & |x| > \epsilon \end{cases}$

$$\frac{\sin s\epsilon}{s\epsilon}$$

Inverse fourier Transform.

$$\mathcal{F}^{-1}\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

Probl:- $f(x) = \begin{cases} 1-x^2 & \text{in } |x| \le 1 \\ 0 & \text{in } |x| > 1 \end{cases}$

Hence prove that $\displaystyle\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\, ds = \frac{3\pi}{16}$.

Soln

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1} f(x)\, e^{isx}dx + \int_{-1}^{1} f(x)\, e^{isx}dx + \int_{1}^{\infty} f(x)\, e^{isx}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1}(1-x^2)\, e^{isx}dx\right] \quad x^2$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{isx}dx - \int_{-1}^{1} x^2 e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-1}^{1} - \left[e^{isx}(x^2-2x+1)\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{is}}{is} - \frac{e^{-is}}{is}\right] - \left[e^{is}(1-2+1) - e^{-is}(1+2+1)\right]$$

$x e^{-x} = e^{x}(x-1)+c$

$x^2 e^{-x} = -e^{-x}(x^2-2x+1)+c$

$u = 1 - x^2 \quad dv = e^{isx}$

$u' = -2x \quad v = \frac{e^{isx}}{is}$

$u'' = -2 \quad v_1 = \frac{e^{isx}}{-s^2}$

$u''' = 0. \quad v_2 = \frac{e^{isx}}{-is^3}$

$v_3 = \frac{e^{isx}}{s^4}$

$$\frac{e^{is} - e^{-is}}{2} - \frac{s e^{is} - s e^{-is}}{2}$$

$$= \left[\frac{4(\sin s - s\cos s)}{s^3}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-x^2)\frac{e^{isx}}{is} - 2x\left(\frac{e^{isx}}{-s^2}\right) + -2\left(\frac{e^{isx}}{is^3}\right)\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(0 + \frac{2e^{is}}{s^2} - \frac{2e^{is}}{is^3}\right) - \left(0 + \frac{2e^{-is}}{s^2} - \frac{2e^{-is}}{is^3}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left(\frac{-2e^{is} - 2e^{-is}}{s^2}\right) + \left(\frac{-2e^{is} + 2e^{-is}}{is^3}\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{-2\left(\frac{e^{is} + e^{-is}}{s^2}\right) - 2\left(\frac{e^{is} - e^{-is}}{is^3}\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2i(s\cos s - 2\sin s)}{is^3}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{4i(\sin s - s\cos s)}{is^3}\right]$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\left[\frac{4(\sin s - s\cos x)}{s^3}\right]$$

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by using Fourier Inverse transform :-

$$\cancel{\mathcal{F}^{-1}\mathcal{F}}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\left(4\,\frac{(\sin s - s\cos s)}{s^3}\right)e^{-isx}\, ds$$

$$= \frac{4}{2\pi}\int_{-\infty}^{\infty}\frac{(\sin s - s\cos s)(\cos sx - i\sin sx)}{s^3}\, ds$$

$$= 2/\pi\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\, ds - \int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\sin sx\, ds$$

$$= 2/\pi \times 2\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\, ds$$

$$f(x) = 4/\pi\left[\int_0^{\infty}\frac{\sin s\cos sx}{s^3}\, ds - \int_0^{\infty}\frac{s\cos s\cos sx}{s^3}\, ds\right]$$

$$\int_0^{\infty}\frac{\sin s - s\cos s}{s^3}\cos sx\, ds = \pi/4\; f(x)$$

Put $x = 1/2$

$$\int_0^{\infty}\frac{\sin s - s\cos s}{s^3}\cos s/2\, ds = \pi/4\,(1-x^2)$$

$$= \pi/4\,(3/4)$$

$$= \boxed{3\pi/16.}$$

(12) $f(x) = x e^{-x} \quad 0 \le x \le \infty$ (22)

Soln

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-x(1-is)}\, dx$$

$u = x \qquad dv = e^{-x(1-is)}$

$u' = 1 \qquad V_1 = \frac{e^{-x(1-is)}}{-(1-is)}$

$V_2 = \frac{e^{-x(1-is)}}{(1-is)^2}$

$$= \frac{1}{\sqrt{2\pi}} \left[u\left(\frac{e^{-x(1-is)}}{-(1-is)}\right) - \frac{e^{-x(1-is)}}{(1-is)^2} \right]_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\infty \left(\frac{e^{-\infty(1-is)}}{-(1-is)}\right) - \frac{e^{-x(1-is)}}{(1-is)^2} \right] - \left[-0 \left(\frac{e^{0(1-is)}}{-(1-is)}\right) - \left(\frac{e^{0(1-is)}}{(1-is)^2}\right) \right] \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(1-is)^2} \right]$$

(13) $f(x) = \begin{cases} \dfrac{\sqrt{2\pi}}{2\epsilon} & |x| \le \epsilon \\ 0 & |x| > \epsilon \end{cases}$

Soln

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-\varepsilon} f(x)\, e^{isx} dx + \int_{-\varepsilon}^{\varepsilon} f(x)\, e^{isx} dx + \int_{\varepsilon}^{\infty} f(x)\, e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{2\pi}}{2\varepsilon}\, e^{isx} dx \right]$$

$$= \frac{1}{\cancel{\sqrt{2\pi}}} \frac{\cancel{\sqrt{2\pi}}}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} e^{isx}\, dx$$

$$= \frac{1}{2\varepsilon} \left[\frac{e^{isx}}{is} \right]_{-\varepsilon}^{\varepsilon}$$

(2) Find Fourier transform of

$$f(x) = \begin{cases} a-|x| , & |x|<a \\ 0 & |x|>a>0 \end{cases}$$

Solⁿ: deduce that $\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \pi/2$

$$F f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} f(x) e^{isx} dx + \int_{-a}^{a} f(x) e^{isx} dx + \int_{a}^{\infty} f(x)\, e^{isx} dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} a-|x|\, e^{isx} dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\cancel{\int_{-a}^{a}}\right]$$

$u = a-|x|$ $\quad dv = e^{isx}$

$u' = \pm 1$ $\quad v_1 = e^{isx}/is$

$u'' = 0$ $\quad v_2 = \frac{e^{isx}}{-s^2}$

$$= \frac{1}{\sqrt{2\pi}}\left[a-|x|\left(\frac{e^{isx}}{is}\right) - (-1)\frac{e^{isx}}{-s^2}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left((0) - \frac{e^{isa}}{s^2}\right) - (0) + \frac{e^{isa}}{s^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} - \left(\frac{e^{isa} - e^{-isa}}{s^2}\right)$$

$$\frac{1}{\sqrt{2\pi}} \quad \frac{-2i\sin sa}{s^2} \qquad \frac{\sqrt{2}}{\pi} \Big/ s^2$$

(25)

$=$

$\frac{\sin t}{t}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)\, e^{-isx}\, ds.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sqrt{2}}{\pi} \left(\frac{1-\cos as}{s^2}\right) e^{-isx}\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-\cos as}{s^2}\, e^{-isx}\, ds.$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2\sin^2 as/2}{s^2}\, e^{isx}\, ds.$$

$$= 2/\pi \int_{-\infty}^{\infty} \left(\frac{\sin as/2}{s}\right)^2 e^{isx}\, ds.$$

~~put $t = as/2$~~ $x = 0$; $a = 2$.

$$= 2/\pi \int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)^2 e^{0}\, ds$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = \pi/2\, f(x)$$

$$= \pi/2 \{a - |x|\}$$

$$= \pi/2\, (2)$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds. \qquad = \pi$$

$$2 \int_{0}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = \pi \therefore = \pi/2.$$

$$\int_{0}^{\infty} \left(\sin t/t\right)^2 dt = \pi/2 \qquad (\because s \text{ is dummy variable})$$

(4) Find the Fourier transform of (28)

$$f(x) = e^{-|x|}.$$

Soln:

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|}\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{-|x|} e^{isx}\, dx + \int_{0}^{\infty} e^{-|x|} e^{isx}\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{x} e^{isx}\, dx + \int_{0}^{\infty} e^{-x} e^{isx}\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{x(1+is)}\, dx + \int_{0}^{\infty} e^{-x(1-is)}\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{e^{x(1+is)}}{(1+is)} \right]_{-\infty}^{0} + \left[\frac{e^{-x(1-is)}}{-(1-is)} \right]_{0}^{\infty} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{1}{(1+is)} - (0) \right] + \left[(0) + \frac{1}{1-is} \right] \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{(1-is) + 1 + is}{(1+is)(1-is)} \right)$$

$(1+is) + (1+is)$

$1 - is - 1 - is$

$-2is$

$$= \frac{1}{\sqrt{2\pi}} \frac{\cancel{-2is}\,2}{1+s^2} = \sqrt{2/\pi} \left(\frac{1}{1+s^2} \right)$$

(5) Find the Fourier transform of $f(x) = e^{-|s|y}$ (29)

Inverse

Soln:

$$\cancel{F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

$$\cancel{= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|s|y}\, e^{isx}\, dx}$$

$$f(x) = e^{-|s|y}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|s|y}\, e^{-isx}\, ds.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{-|s|y}\, e^{-isx}\, ds + \int_{0}^{\infty} e^{-|s|y}\, e^{-isx}\, ds \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{sy}\, e^{-isx}\, ds + \int_{0}^{\infty} e^{-sy}\, e^{-isx}\, ds \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{s(y-ix)}\, ds + \int_{0}^{\infty} e^{-s(y+ix)}\, ds \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{e^{s(y-ix)}}{(y-ix)} \right]_{-\infty}^{0} + \left[\frac{e^{-s(y+ix)}}{-(y+ix)} \right]_{0}^{\infty} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{1}{(y-ix)} - (0) \right] + \left[0 + \frac{1}{y+ix} \right] \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{y+\cancel{ix}+y-\cancel{ix}}{y^2+x^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \, {}^{2y}\!/_{y^2+x^2}$$

$$= \sqrt{2/\pi} \left(y/_{y^2+x^2} \right)$$

Properties.

① Linear property :-

$$F[a\,f(x) \pm b\,g(x)] = a\,F(s) \pm b\,G(s)$$

Pf :- We know that

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$F[a\,f(x) \pm b\,g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [a\,f(x) \pm b\,g(x)]\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a\,f(x)\, e^{isx}\, dx \pm \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} b\,g(x)\, e^{isx}\, dx.$$

$$= a \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \right] \pm b \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx}\, dx \right]$$

$$= a\,F(s) \pm b\,G(s)$$

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2. Change of scale property:-

$$F[f(ax)] = \frac{1}{a} F\left(s/a\right), \quad a>0$$

Proof:-

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

Put $ax = y$ $\qquad x=-\infty \Rightarrow y$

$a\,dx = dy$ $\qquad x=\infty \Rightarrow$

$dx = dy/a$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{isy/a}\, dy/a$$

$$= \frac{1}{a}\left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{i s/a\, y}\, dy\right\}$$

$$= \frac{1}{a} F\left(s/a\right)$$

3. Shifting theorem (or) property:-

(i) $F[f(x-a)] = e^{ias} F(s)$

(ii) $F[e^{iax} f(x)] = F(s+a)$

Proof

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

$$x - a = y$$

$$dx = dy$$

$$x = -\infty \Rightarrow y = -\infty$$

$$x = \infty \Rightarrow y = \infty$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{is(y+a)}\, dy.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{ias + isy}\, dy.$$

$$= \frac{e^{ias}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{isy}\, dy.$$

$$= e^{ias} \times \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{isy}\, dy \right\}$$

$$= e^{ias} \cdot F(s) //$$

(ii) $F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx.$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ix(s+a)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} F(s+a)$$

Note:- $F[e^{-iax} f x] = F(s-a)$

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(4). Modulation property:-

$$F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$$

Proof:-

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax\; e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right) e^{isx}\,dx.$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty} f(x)\, e^{iax} e^{isx}\,dx + \int_{-\infty}^{\infty} f(x)\, e^{-iax} e^{isx}\,dx\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{ix(s+a)}\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{ix(s-a)}\,dx\right\}$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right] //$$

5) $F[x^n f(x)] = (-i)^n \frac{d^n F(s)}{ds^n}$

Proof:-

$$F(s) = F(f(x))$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx.$$

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$$\frac{d}{ds} F(s) = \frac{d}{ds}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \partial/\partial x\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} (ix)\, dx.$$

$$\frac{d^2}{ds^2} F(s) = \frac{d}{ds}\left(d/ds\, F(s)\right)$$

$$= d/ds \left[1/\sqrt{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{isx} (ix)\, dx\right]$$

$$= 1/\sqrt{2\pi} \int_{-\infty}^{\infty} f(x)\, \partial/\partial s\, e^{isx} (ix)\, dx$$

$$= 1/\sqrt{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{isx} (ix^2)\, dx$$

In generally,

$$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, (ix)^n\, dx.$$

$$= (i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underline{x^n f(x)}\, e^{isx}\, dx$$

$$= i^n \cancel{1/\sqrt{\ }}\; F\left[x^n f(x)\right]$$

$$(-i)^n \frac{d^n}{ds^n} F(s) = F\left[x^n f(x)\right]$$

6) $F[f'(x)] = -is F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

Proof:

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$$

$$\mathcal{F}[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f'(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\{f(x)\}\, e^{isx} dx$$

$u = e^{isx}$ $\quad dv = d f(x)$

$du = e^{isx}(is)$ $\quad v = f(x)$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[e^{isx} f(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\, e^{isx}\, is\, dx \right.$$

$$= \frac{1}{\sqrt{2\pi}} \left[(0 - 0) - is \int_{-\infty}^{\infty} f(x)\, e^{isx} dx \right]$$

$$= (-is) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$$

$$F[f'(x)] = (-is)\, \mathcal{F}(s)$$

$$F\left[\frac{d}{dx} f(x) \right] = (-1)\, F(s)$$

$$F\left[\frac{d^n}{dx^n} f(x) \right] = (-is)^n F(s) \;//$$

$$\Rightarrow = F\left[f^n(x) \right] //$$

(7) $F\left[\int_a^x f(x)\,dx\right] = \dfrac{F(s)}{(-is)}$

Pf: Let $\phi(x) = \int_a^x f(x)\,dx$

$$\phi'(x) = f(x)$$

We know that

$$F[f'(x)] = -is\,F(s)$$

$$F[\phi'(x)] = -is\,F[\phi(x)]$$

$$= -is\,F\left[\int_a^x f(x)\,dx\right]$$

$$F\left[\int_a^x f(x)\,dx\right] = \frac{F[\phi' x]}{-is}$$

$$= \frac{1}{-is} F[f(x)]$$

$$= \frac{F(s)}{-is} //$$

(8) $F[\overline{f(x)}] = \overline{F(-s)}$

Proof

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\,dx.$$

taking Complex Congigate On bothsides We get

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$$\overline{F(-s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{isx}\, dx$$

$$= \mathcal{F}\left[\left(\overline{f(x)}\right)\right]$$

$$\overline{\mathcal{F}(-s)} = \mathcal{F}\left[\overline{f(x)}\right]$$

(9) $F\left[f(-x)\right] = F(-s)$

$$\mathcal{F}[(f(x))] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$\mathcal{F}\left[f(-x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x)\, e^{isx}\, dx$$

Put $x = -y$ | $x = -\infty \Rightarrow y = \infty$

$dx = -dy$ | $x = \infty \Rightarrow y = -\infty$

$$F\left[f(-x)\right] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(y)\, e^{-isy}\, (-dy)$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(y)\, e^{-isy}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-isy}\, dy.$$

$$= F(-s)$$

(10) $F\left[\overline{f(-x)}\right] = \overline{F(s)}$

$$\mathcal{F}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \quad \text{Complex Congical}$$

$$\overline{\mathcal{F}(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{-isx}\, dx.$$

Put $x = -y$ | $x = -\infty \quad y = \infty$

$dx = -dy$ | $x = \infty \quad y = -\infty$

$$\overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \overline{f(-y)}\, e^{isy} (-dy)$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\int_{\infty}^{-\infty} \overline{f(-y)}\, e^{isy} dy \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-y)}\, e^{isy} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-x)}\, e^{isx} dx$$

($\because$ y is the dummy varia)

$$= F\left[\overline{f(-x)}\right]$$

$$\therefore F\left[f(-x)\right] = \overline{F(s)}$$

Prob: If the Fourier transform of $f(x)$ is $\sqrt{2/\pi}\,\frac{\sin as}{s}$ find $F\left[f(x)\left(1+\cos \pi x/a\right)\right]$

Soln:-

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\sqrt{2/\pi}\,\frac{\sin as}{s} = F(s) \quad \text{--- (1)}$$

We know that Modulation theorem

$$F[f(x)\cos ax] = \frac{1}{2}\left[F(s+a)+F(s-a)\right]$$

$$F\left[f(x)\cos \frac{\pi x}{a}\right] = \frac{1}{2}\left\{\left[\sqrt{2/\pi}\,\frac{\sin a(s+\pi/a)}{s+\pi/a}\right] + \left[\sqrt{2/\pi}\,\frac{\sin a(s-\pi/a)}{s-\pi/a}\right]\right\}$$

$$= \frac{1}{2}\left\{\sqrt{2/\pi}\,\frac{\sin a\left(\frac{as+\pi}{a}\right)}{(as+\pi/a)} + \sqrt{2/\pi}\,\frac{\sin a\left(as-\pi/a\right)}{(as-\pi/a)}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{a\sin(as+\pi)}{as+\pi} + \frac{\sin(as-\pi)}{as-\pi}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{-a\sin as}{as+\pi} + \frac{a\sin(\pi-as)}{as-\pi}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-a\sin as}{as+\pi} + \frac{a}{as-\pi}(-\sin as)\right]$$

$$= -\frac{a\sin as}{\sqrt{2\pi}}\left[\frac{1}{as+\pi} + \frac{1}{as+\pi}\right]$$ $\sin(180+\theta) = -\sin\theta$, $\sin(180-\theta) = \sin\theta$

(4)

$$= \frac{a\sin as}{\sqrt{2\pi}}\left[\frac{as-\pi+as+\pi}{(as+\pi)(as-\pi)}\right]$$

$$= -\frac{a\sin as}{\sqrt{2\pi}}\left[\frac{2as}{a^2s^2-\pi^2}\right]$$

$$= -\frac{a^2 s\sin as\sqrt{2}}{\sqrt{\pi}\,a^2s^2-\pi^2}$$

$$= -\sqrt{2/\pi}\,\frac{a^2 s\sin as}{a^2s^2-\pi^2} \quad \text{——(6)}$$

adding (5) & (6)

$$F[f(x)] = \sqrt{2/\pi}\,\frac{\sin as}{s} = F(s)$$

——(7)

(Part a) ; add

$$F[f(x)] + F[f(x)\cos \pi x/a]$$

$$= \sqrt{2/\pi}\,\frac{\sin as}{s} - \sqrt{2/\pi}\,\frac{a^2 s\sin as}{a^2s^2-\pi^2}$$

$$F[f(x)(1+\cos\pi x/a)] = \sqrt{2/\pi}\,\sin as\left[\frac{1}{s} - \frac{a^2 s}{a^2s^2-\pi^2}\right]$$

(3) Find fourier transform of $e^{-a|x|}$ (12)

deduce that $F[x\, e^{-a|x|}] = (i)\sqrt{2/\pi}\, \frac{2as}{(s^2+a^2)^2}$

Soln

We know that

$$F[f(x)] = F[e^{-a|x|}]$$

$$= \sqrt{2/\pi}\left(\frac{a}{s^2+a^2}\right) = F(s)$$

$$F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n}(F(s))$$

Here $n = 1$

$$F[x\, f(x)] = (-i)^n \frac{d}{ds} F(s)$$

$$= -i\, \frac{d}{ds}\left[\sqrt{2/\pi}\left(\frac{a}{s^2+a^2}\right)\right]$$

$$= -i\, \sqrt{2/\pi}\, a(-1)(s^2+a^2)^{-2} \cdot 2s$$

$$= (-i)\sqrt{2/\pi}\, \frac{-2as}{(s^2+a^2)^2}$$

$$= i\sqrt{2/\pi}\, \frac{2as}{(s^2+a^2)^2}$$

(4) $f(x) = e^{-|x|}$ deduce $e^{-|x|}\cos 2x$

Soln $F[f(x)] = \sqrt{2/\pi}\left(\frac{1}{1+s^2}\right) = F(s)$

$F[e^{-|x|}\cos 2x] = \cancel{F}[$

W.K.that

$F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$

$F[e^{-|x|}\cos 2x] = \frac{1}{2}\left[\sqrt{2/\pi}\,\frac{1}{(s+2)^2+1} + \sqrt{2/\pi}\,\frac{1}{(s-2)^2+1}\right]$

$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{(s+2)^2+1} + \frac{1}{(s-2)^2+1}\right]$

$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{s^2+4+4s+1} + \frac{1}{s^2+4-4s+1}\right]$

$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{s^2+4s+5} + \frac{1}{s^2-4s+5}\right]$

$= \frac{1}{\sqrt{2\pi}}\left[\frac{s^2 - \cancel{4s}+5+s^2+\cancel{4s}+5}{(s^2+4s+5)(s^2-4s+5)}\right]$

$= \sqrt{1/2\pi}\left[\frac{2s^2+10}{(s^2+4s+5)(s^2-4s+5)}\right]$

Convolution of Two Functions.

The Convolution of $f(x)$ & $g(x)$ is definded as $(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(x-t)\,dt$

$f*g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(x-t)\,dt.$

Convolution theorem for Fourier Transform

(1) If $F(s)$ and $G(s)$ are the fourier transform of $f(x)$ & $g(x)$ respectively then, the fourier transform of the convolution of $f(x)$ & $g(x)$ is the Product of the fourier transform. i.e)

$$F[(f*g)(x)] = F(s) \cdot G(s)$$

Proof:-

$$F[(f*g)(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)(x)\, e^{isx} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\, e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)\, e^{isx} dx \right] dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left[F(g(x-t)) \right] dt$$

$$\left\{ \begin{array}{l} F[g(x-t)] = e^{its} G(s) \longrightarrow \text{Shifting theorem.} \\ F[f(x-a)] = e^{ias} F(s) \end{array} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{its}\, G(s)\, dt$$

$$= \frac{G(s)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{its}\, dt$$

$$= G(S) \cdot \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{its} dt \right\}$$

$$F[(f*g)(x)] = G(S) \cdot F(S)$$

(2) **Parseval's Identity:-**

Let $F(s)$ be the fourier transform of $f(x)$ then $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$.

Proof:-

W.k.that

$$F[(f*g)(x)] = F(s) \cdot G(s)$$

$$(f*g)(x) = F^{-1}[F(s) \cdot G(s)]$$

$$F^{-1}[F(s) \cdot G(s)] = f*g$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot G(s)\, e^{-isx} ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Put $x=0$. we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot G(s)\, ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(-t)\, dt \quad \text{--- (1)}$$

$$\text{Let } g(-t) = \overline{f(t)} \quad \text{--- (2)}$$

$$g(t) = \overline{f(-t)} \quad \text{--- (3)}$$

$$F\, g(t) = F[\overline{f(-t)}]$$

$$G(S) = F(\overline{f(-x)})$$

(4)

$$G(S) = \overline{F(S)}$$

Substituting (2) & (4) in (1)

$$\int_{-\infty}^{\infty} F(s)\,\overline{F(s)}\,ds = \int_{-\infty}^{\infty} f(t)\,\overline{f(t)}\,dt$$

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

$$\left[\because F(s)\,\overline{F(s)} = |F(s)|^2\right]$$

Hence proved.

Exercise

Find fourier transform of $f(x)$

$$f(x) = \begin{cases} 1 & , |x| < a \quad -a < x < a. \\ 0 & , |x| > a > 0. \end{cases}$$

(i) The fourier transform of

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-a} f(x)\, e^{isx} dx + \int_{-a}^{a} f(x)\, e^{isx} dx + \int_{a}^{\infty} f(x)\, e^{isx} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-a}^{a} 1\, e^{isx} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isa} - e^{-isa}}{is} \right]$$

2020.12.

$$= \frac{1}{\sqrt{2\pi}} \frac{2i\sin sa}{is}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2\sin sa}{s}$$

$$= \sqrt{2/\pi} \frac{\sin sa}{s}$$

(ii)

The Inverse transform of

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2/\pi} \frac{\sin sa}{s} e^{-isx} ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa}{s} (\cos sx - i\sin sx) ds$$

$$= \frac{1}{\pi} \left\{ \int_{-\infty}^{\infty} \frac{\sin sa}{s} \cos sx \, ds - \int_{-\infty}^{\infty} \frac{\sin sa}{s} i\sin sx \, ds \right\}$$

$$= \frac{1}{\pi} \left\{ \int_{-\infty}^{\infty} \frac{\sin sa}{s} \cos sx \, ds \right\}$$

$$= 2/\pi \int_{0}^{\infty} \frac{\sin sa}{s} \cos x \, ds$$

put $x = 0$

$$= 2/\pi \int_{0}^{\infty} \frac{\sin sa}{s} ds$$

$$= \int_{0}^{\infty} \frac{\sin sa}{s} ds = \pi/2 \, f(0)$$

Put $as = t$ $\quad s = 0 \Rightarrow t = 0$

$s = t/a$ $\quad s = \infty \Rightarrow t = 0$

$ds = dt/a$

$$= \int_0^\infty \frac{\sin t}{t/a} \, dt/a = \pi/2 \quad (SS)$$

$$= \int_0^\infty \frac{\sin t}{t} dt = \pi/2 \,//$$

(ii) The parseval's identity is

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

We have,

$$\int_{-a}^{a} (1)^2 dx = \int_{-\infty}^{\infty} \left(\sqrt{2/\pi} \cdot \frac{\sin as}{s}\right)^2 ds$$

[using (1)]

$\because f(x) = 1$ in $(-a, a)$

$f(x) = 0$ $(-\infty, -a)$ and (a, ∞)

$$(x)_{-a}^{a} = 2/\pi \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds.$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds = \pi/2 \, (a + a)$$

$$= \pi/2 (2a)$$

$$= \pi a$$

$$2 \int_0^\infty \left(\frac{\sin as}{s}\right)^2 = \pi a$$

Put $as = t$

$s = t/a$

$ds = dt/a$

$$\int_0^\infty \left(\frac{\sin t}{t/a}\right)^2 \frac{dt}{a} = \pi a/2$$

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi a}{2} \times \frac{a}{a^2}$$

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2.$$

(2) Find the fourier transform of

$$f(x) = \begin{cases} 1-|x| & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

deduce that

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$$

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \pi/3.$$

Soln:

(i) The fourier transform of

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-1} f(x) e^{isx} dx + \int_{-1}^{1} [1-|x|]\, e^{isx} dx + \int_{1}^{\infty} f(x)\, e^{isx} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} [1-|x|] [\cos sx + i \sin sx] \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx \, dx \quad \int_{-1}^{1} (1-|x|) \, i \sin s \ldots$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \, 2 \int_{0}^{1} (1-x) \cos sx \, dx \qquad \left[\because 1-|x| \ldots \right.$$

$$= \sqrt{2/\pi} \int_{0}^{1} (1-x) \cos sx \, dx$$

$u = 1-x \qquad dv = \cos sx$

$u' = -1 \qquad v = \frac{\sin sx}{s}$

$\qquad\qquad v_1 = -\frac{\cos sx}{s^2}$

$$= \sqrt{2/\pi} \left[(1-x) \frac{\sin sx}{s} + \frac{-\cos sx}{s^2} \right]_{0}^{1}$$

$$= \sqrt{2/\pi} \left[\left((0) - \frac{\cos s}{s^2} \right) - \left((0) - 1/s^2 \right) \right]$$

$$= \sqrt{2/\pi} \left[-\frac{\cos s}{s^2} + \frac{1}{s^2} \right]$$

$$F(s) = \sqrt{2/\pi} \left[\frac{1-\cos s}{s^2} \right]$$

(ii) The inverse transform of

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2/\pi} \left(\frac{1-\cos s}{s^2} \right) e^{-isx}\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1-\cos s}{s^2} \left[\cos sx - i \sin sx \right]$$

$$= \frac{1}{\pi} \left\{ \int_{-\infty}^{\infty} \frac{1-\cos s}{s^2} \cos sx\, ds - i \int_{-\infty}^{\infty} \frac{1-\cos s}{s^2} \sin sx \right.$$

$$= \frac{1}{\pi}\, 2 \int_{0}^{\infty} \left(\frac{1-\cos s}{s^2} \right) \cos sx\, ds$$

Put $x = 0$

$$\int_{0}^{\infty} \frac{1-\cos s}{s^2}\, ds = \pi/2.$$

$$\int_{0}^{\infty} \frac{2 \sin^2 s/2}{s^2}\, ds = \pi/2$$

Put $s/2 = t$

$$ds/2 = dt$$

$$\int_{0}^{\infty} \frac{2 \sin^2 t}{4t^2}\, 2dt = \pi/2.$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \pi/2.$$

(iii) using parseval's identity

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos s}{s^4}\right)^2 ds = \int_{-1}^{1} (1-|x|)^2 dx$$

$f(x) = 0$ in $(-\infty$

and $(1, \infty)$

$$= 2\int_0^1 (1-x)^2 dx.$$

$|x| = x \quad x>0$

$1-|x| \to$ even

$$= 2\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= -2/3\,[0-1]$$

$$= 2/3$$

$$2/\pi \int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4} ds = 2/3$$

$$\int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4} ds = \pi/3$$

Put $s = 2t$

$ds = 2dt$

$$\int_{-\infty}^{\infty} \frac{(1-\cos 2t)^2}{16t^4}\, 2\,dt = \pi/3$$

$$\int_{-\infty}^{\infty} \frac{(1-\cos 2t)^2}{8t^4}\, dt = \pi/3$$

$$2\int_{0}^{\infty} \frac{(1-\cos 2t)^2}{8t^4}\, dt = \pi/3$$

$$\int_{0}^{\infty} \frac{(2\sin^2 t)^2}{4t^4}\, dt = \pi/3$$

$\frac{(1-\cos 2t)^2}{8t} \Rightarrow$ even

$\sin^2 x = \frac{1-\cos 2x}{2}$

$$\int_{0}^{\infty} \frac{\sin^4 t}{t^4}\, dt = \pi/3$$

$$\int_{0}^{\infty} \frac{\sin^4 t}{t^4}\, dt = \pi/3$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \pi/3$$

Eg ③ Show that the fourier transform of

$$f(x) = \begin{cases} a^2 - x^2 & |x| < a \\ 0 & |x| > a > 0 \end{cases}$$ is $2\sqrt{2/\pi}\left(\frac{\sin as - as\cos as}{s^3}\right)$

Hence deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt = \pi/4$.

Soln: using parseval's identity show that $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \pi/15$.

Soln :-

(i) The fourier transform of

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-a} f(x)\, e^{isx} dx + \int_{-a}^{a} f(x)\, e^{isx} dx + \int_{a}^{\infty} f(x)\, e^{isx} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} (a^2 - x^2)\, e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} (a^2 - x^2) \cos sx\, dx + \int_{-a}^{a} (a^2 - x^2)\, i \sin sx \right.$$

$$= \frac{2 \cdot 1}{\sqrt{2\pi}} \int_{0}^{a} (a^2 - x^2) \cos sx\, dx$$

$u = a^2 - x^2$ $\quad$ $dv = \cos sx$

$u' = -2x$ $\quad$ $v = \frac{\sin sx}{s}$

$u'' = -2$ $\quad$ $v_1 = -\frac{\cos sx}{s^2}$

$u''' = 0$ $\quad$ $v_3 = -\frac{\sin sx}{s^3}$

$$= \sqrt{2/\pi} \left[(a^2 - x^2) \frac{\sin sx}{s} - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right) \right]_0^a$$

$$= \sqrt{2/\pi} \left[(a^2 - x^2) \frac{\sin sx}{s} - (2x) \frac{\cos sx}{s^2} + 2\frac{\sin sx}{s^3} \right]_0^a$$

$$= \sqrt{2/\pi}\left[\left(0 - 2a\frac{\cos sa}{s^2} + \frac{2\sin sa}{s^3}\right) - \left(0 - 0\right)\right]$$

$$= \sqrt{2/\pi}\left[\frac{2\sin sa - 2as\cos sa}{s^3}\right]$$

$$= 2\sqrt{2/\pi}\left[\frac{\sin sa - as\cos sa}{s^3}\right]$$

(ii) The Inverse transform of $f(x)$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{2/\pi}\left[\frac{\sin sa - as\cos sa}{s^3}\right] e^{-isx} ds$$

$$= \frac{2}{\pi}\left\{\int_{-\infty}^{\infty}\frac{\sin sa - as\cos sa}{s^3}\cos sx\, ds + \int_{-\infty}^{\infty}\frac{\sin sa - as\cos as}{s^3}\, i\sin sx\, ds\right.$$

$$f(a^2 - x^2) = \frac{2\cdot 2}{\pi}\int_{-\infty}^{\infty}\frac{\sin sa - as\cos sa}{s^3}\cos sx\, ds \rightarrow$$

put $as = t$ $\quad ads = dt$ $\quad$ put $x = 0$

$s = t/a$ $\quad ds = dt/a$

$s^2/s^2 t^2/a^2$

$= 4/\pi \int^{\infty} \sin t$

$$= \sqrt{2/\pi}\left[\left(0 - 2a\frac{\cos sa}{s^2} + \frac{2\sin sa}{s^3}\right) - (0 - 0)\right]$$

$$= \sqrt{2/\pi}\left[\frac{2\sin sa - 2as\cos sa}{s^3}\right]$$

$$= 2\sqrt{2/\pi}\left[\frac{\sin sa - as\cos sa}{s^3}\right]$$

(ii)

The Inverse transform of $f(x)$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{2/\pi}\left[\frac{\sin sa - as\cos sa}{s^3}\right] e^{-isx}\, ds$$

$$= \frac{2}{\pi}\left\{\int_{-\infty}^{\infty}\frac{\sin sa - as\cos sa}{s^3}\cos sx\, ds + \int_{-\infty}^{\infty}\frac{\sin sa - as\cos as}{s^3}\, i\sin sx\, ds\right\}$$

$$f(a^2-x^2) = \frac{2\cdot 2}{\pi}\int_{0}^{\infty}\frac{\sin sa - as\cos sa}{s^3}\cos sx\, ds \longrightarrow \text{even}$$

Put $as = t$ $\quad ads = dt$ $\quad$ put $x = 0$

$s = t/a \quad ds = dt/a$

$s^2 \; t^2/a^2$

$$= 4/\pi \int_0^{\infty} \sin t$$

$$a^2 - x^2 = 4/\pi \int_0^\infty \frac{\sin t - t\cos t}{(t/a)^3} \cos t/a\, x \, dt/a$$

put $x = 0$ $a = 1$

$$1 = 4/\pi \int_0^\infty \frac{\sin t - t\cos t}{t^3} \, dt$$

$$\Rightarrow \int_0^\infty \frac{\sin t - t\cos t}{t^3} \, dt = \pi/4.$$

(iii) using parsevals identity

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

[using A]

$$\int_{-\infty}^{\infty} \left[2\sqrt{2/\pi} \frac{(\sin s - s\cos s)}{s^3}\right]^2 ds = \int_{-1}^{1} (1-x^2)^2 dx.$$

[put $a=1$ in the given func]

$$\frac{8}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_0^1 (1-x^2)^2 dx.$$

$$\frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_0^1 (1+x^4-2x^2)$$

$$= 2\left[x + \frac{x^5}{5} - 2 \cdot \frac{x^3}{3}\right]_0^1$$

$$= 2\left[(1 + 1/5 - 2/3) - 0\right]$$

$$= 2\left[\frac{15+3-10}{15}\right] = 2 \cdot 8/15$$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \pi/15$$

(or)

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \pi/15$$ Replacing s by t

(4) Page no: 26 → Fourier transform $e^{-a|x|}$ if $a>0$,
Find the fourier transform of $e^{-a|x|}$ if $a>0$.
Deduce that $\int_0^\infty \frac{1}{(x^2+a^2)^2} dx = \pi/4a^3$

using parsavals identity.

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\int_{-\infty}^{\infty} \left[e^{-a|x|}\right]^2 dx = \int_{-\infty}^{\infty} \left(\sqrt{2/\pi}\, \frac{a}{s^2+a^2}\right)^2 ds.$$

$$2\int_0^\infty (e^{-ax})^2 dx = 2\int_0^\infty 2/\pi \left(\frac{a^2}{(s^2+a^2)^2}\right) ds$$

$\because e^{-a|x|} = e^{-ax}$ in $(0,\infty)$

$$\int_0^\infty e^{-2ax} dx = \frac{2}{\pi}\int_0^\infty \frac{a^2}{(s^2+a^2)^2} ds$$

$\left(\sqrt{2/\pi}\, \frac{a}{s^2+a^2}\right) \to$ even

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2a^2}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)^2} ds.$$

$$\frac{e^{-\infty}}{2a} - \frac{e^{0}}{-2a} = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2}$$

$$0 + \frac{1}{2a} = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2}$$

$$\int_0^\infty \frac{ds}{(s^2+a^2)^2} = \frac{\pi}{4a^3}, \quad a>0$$

(or)

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \pi/4a^3 \,//$$

(5) Find the fourier transform of $e^{-|x|}$ deduce that $\int_0^\infty \frac{dx}{(x^2+1)^2} = \pi/4$.

Soln:- fourier transform - P No. 28

$$F[e^{-|x|}] = \sqrt{2/\pi}\left(\frac{1}{s^2+1}\right)$$

(Replace a=1 in eq 4)

using parseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-\infty}^{\infty} |e^{-|x|}|^2 dx = \int_{-\infty}^{\infty}\left(\sqrt{2/\pi}\,\frac{1}{s^2+1}\right)^2 ds$$

$$e^{-|x|} = e^{-x} \text{ in } (0,\infty)$$

$$e^{-|x|} = e^{x} \text{ in } (-\infty,0)$$

$$\int_{-\infty}^{0} (e^{x})^2 dx + \int_{0}^{\infty} (e^{-x})^2 dx = 2\int_{0}^{\infty} \frac{2}{\pi}\left(\frac{1}{s^2+1}\right)^2 ds$$

$$\int_{-\infty}^{0} e^{2x} dx + \int_{0}^{\infty} e^{-2x} dx = \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{1}{s^2+1}\right)^2 ds$$

$$\left(\frac{e^{2x}}{2}\right)_{-\infty}^{0} + \left(\frac{e^{-2x}}{-2}\right)_{0}^{\infty} = \frac{4}{\pi}\int_{0}^{\infty}\frac{1}{(s^2+1)^2} ds$$

$$\left(\frac{e^{0}}{2} - \frac{e^{-\infty}}{2}\right) + \left(\frac{e^{-\infty}}{-2} - \frac{e^{0}}{-2}\right) = 4/\pi \int_{0}^{\infty} \frac{ds}{(s^2+1)^2}$$

$$\left(\frac{1}{2} - 0\right) + \left(0 + 1/2\right) = \frac{4}{\pi}\int_{0}^{\infty}\frac{1}{(s^2+1)^2} ds$$

$$1 = 4/\pi \int_{0}^{\infty}\frac{1}{(s^2+1)^2} ds$$

$$\int_{0}^{\infty}\frac{1}{(s^2+1)^2} ds = \pi/4$$

$$\int_{0}^{\infty}\frac{1}{(x^2+1)^2} dx = \pi/4$$

(b) If $f(x) = \begin{cases} \cos x & |x| < \pi/2 \\ 0 & |x| > \pi/2 \end{cases}$ then.

find the fourier tranform of $f(x)$ and hence evaluate

$$\int_{0}^{\infty} \frac{\cos^2\left(\frac{\pi x}{2}\right)}{(1-x^2)^2} dx$$ using parsevals Identity.

Soln $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\pi/2} 0\, e^{isx} dx + \int_{-\pi/2}^{\pi/2} \cos x\, e^{isx} dx$

$+ \int_{\pi/2}^{\infty} 0\, e^{isx} dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} e^{isx} \cos x\, dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} \cos x\, (\cos sx + i \sin sx)\, dx$

$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\pi/2}^{\pi/2} \cos x \cos sx\, dx + \int_{-\pi/2}^{\pi/2} \cos x\, i \sin sx\, dx \right.$

$\int_{-\pi/2}^{\pi/2} \cos x \cos sx$ is even fun.

$= \frac{1}{\sqrt{2\pi}}\, 2 \int_{0}^{\pi/2} \cos x \cos sx\, dx$

$\left[\cos(A+B) = \cos A \cos B - \sin A \sin B \right.$

$\cos A \cos B = \cos(A+B) + \cos(A-B)$

$= \sqrt{2/\pi} \int_{0}^{\pi/2} \left[\frac{\cos(s+1)x + \cos(s-1)x}{2} \right] dx$

$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\pi/2} [\cos(s+1)x + \cos(s-1)x]\, dx$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1} \right]$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)\pi/2}{s+1} + \frac{\sin(s-1)\pi/2}{s-1} + 0 - 0\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin s\pi/2 \cos\pi/2 + \cos s\pi/2 \sin\pi/2}{s+1} + \frac{\sin s\pi/2 \cos\pi/2 - \cos s\pi/2 \sin\pi/2}{s-1}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\cos s\pi/2}{s+1} - \frac{\cos s\pi/2}{s-1}\right] \quad \because \sin\pi/2 = 1,\ \cos\pi/2 = 0$$

$$= \frac{\cos s\pi/2}{\sqrt{2\pi}}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= \frac{\cos s\pi/2}{\sqrt{2\pi}}\left[\frac{s-1-s-1}{s^2-1}\right]$$

$$= \frac{\cos s\pi/2}{\sqrt{2\pi}}\left(\frac{2}{1-s^2}\right)$$

$$= \sqrt{2/\pi}\left(\frac{\cos s\pi/2}{1-s^2}\right)$$

$$F(s) = \sqrt{2/\pi}\left(\frac{\cos s\pi/2}{1-s^2}\right)$$

Parseval's Identity

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\int_{-\infty}^{\infty}\left[\sqrt{2/\pi}\ \frac{\cos s\pi/2}{1-s^2}\right]^2 ds = \int_{-\pi/2}^{\pi/2} \cos^2 x\, dx$$

$$2/\pi \int_{-\infty}^{\infty} \frac{\cos^2 s\pi/2}{(1-s^2)^2}\, ds = \int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx$$

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$$\frac{4}{\pi}\int_0^\infty \frac{\cos^2 s\pi/2}{(1-s^2)^2}\,ds = 2\int_0^{\pi/2} \frac{(1+\cos 2x)}{2}\,dx$$

$\because 1+\cos 2x \rightarrow$ even

$$= \int_0^{\pi/2} 1+\cos 2x \, dx$$

$$= \left[x + \frac{\sin 2x}{2} \right]_0^{\pi/2}$$

$$= \left[\left(\pi/2 + \sin 2\pi/2\right) - (0+0)\right]$$

$$= \pi/2 + 0 = \pi/2$$

$$\int_0^\infty \frac{\cos^2 s\pi/2}{(1-s^2)^2} = \pi^2/8.$$

(or)

$$\int_0^\infty \frac{\cos^2 \pi x/2}{(1-x^2)^2} = \pi^2/8.$$